

# Reply to Comment on “Time-dependent quasi-Hermitian Hamiltonians and the unitary quantum evolution”

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## Abstract

In his fresh “Comment” (arXiv:0711.0137v1), A. Mostafazadeh reacts on my very recent letter (arXiv:0710.5653v1) where I tried to clarify certain misunderstandings which occurred in A. M., Phys. Lett. B **650**, 208 (2007) [arXiv:0706.1872v2, “Paper”]. As long as the “Comment” offers a new support of the original assertions made in the “Paper”, I feel obliged to re-clarify the matter by extending my argumentation. I insist that it is possible to escape the main conclusion of the “Paper”, indeed. In particular, I point out a gap in the new calculations in “Comment”, add a few remarks on the notation and reconfirm that the unitarity of the time-evolution DOES NOT require the time-independence of the metric operator.

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In the notation used in my preprint [1] as well as in the A. Mostafazadeh’s brand new comment on it [2], the symbol  $\Theta$  denotes a (positive) metric operator with a square root  $\omega := \sqrt{\Theta}$ , both assigned to a possibly time-dependent  $\Theta$ -pseudo-Hermitian Hamiltonian operator  $H$  acting on a reference Hilbert space  $\mathcal{H}$  with the inner product  $\langle \cdot | \cdot \rangle$ . Moreover,

- symbol  $h := \omega H \omega^{-1}$  is chosen to denote the equivalent Hermitian Hamiltonian leading to the evolution operator  $u$  such that  $i\hbar \partial_t u(t) = h(t)u(t)$  and  $u(0) = I$ , where  $I$  stands for the identity operator,

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- the first of Eqs. (17) of [1] (rewritten as Eq. Nr. (3) in [2]) *defines* the auxiliary quantity

$$U_R(t) = \omega(t)^{-1} u(t) \omega(0), \quad (1)$$

- in [2], the subsequent unnumbered equation re-derives Eq. Nr. (11) of ref. [3],

$$\Theta(t) = U_R(t)^{-1\dagger} \Theta(0) U_R^{-1}. \quad (2)$$

On this background the core of the new misunderstandings can be easily spotted as lying in the *incorrect* assumption represented by Eq. Nr. (2) in [2], viz., by the relation

$$i\hbar\partial_t U_R(t) = H(t)U_R(t), \quad U_R(0) = I. \quad (3)$$

An easy explanation of the new puzzle is obtained when we differentiate the definition (1) and reveal that the assumption (3) is manifestly incorrect. At the same time, precisely this contradictory and entirely unfounded relation was postulated in [3] and used to derive the final statement represented by the last Eq. Nr. (4) in ref. [2].

We can repeat that there emerges no obstacle which would violate the unitarity of the quantum evolution when the Hamiltonian  $H = H(t)$  becomes manifestly time-dependent. In [1] we verified that in place of the puzzling Eq. Nr. (12) of ref. [3] (or in place of its unnumbered version preceding eq. Nr. (4) in [2]) one has, simply,

$$H(t)^\dagger = \Theta(t)H(t)\Theta(t)^{-1}. \quad (4)$$

In the other words, one returns to the expected time-dependent quasi-Hermiticity condition for  $H(t)$ .

This being said, it may sound paradoxical when we propose to complement the above brief rebuttal to the Ali Mostafazadeh's *technical* comment by an additional text expressing our *full agreement* with his *philosophical* conclusion that “The root of the misjudgment made in [...] seems to be the rather deceptive nature of the notation”. Let us add a few more remarks on that matter, therefore.

We believe that probably the main source of the possible ambiguities should be seen in the fact that our decision of working with the quasi-Hermitian Hamiltonians  $H$  (i.e., with those which obey the operator identity (4) with a nontrivial  $\Theta \neq I$ ) *implies* that we have to work with *several* Hilbert spaces *at once*.

In a brief detour let us note that in a historical perspective, the work with several Hilbert spaces found its strong, persuasive and *purely physical* original motivation in nuclear physics. In a way reviewed by Scholtz et al [4] people often try to *start* there from a *prohibitively complicated* Hamiltonian operator  $h$  which acts in a standard physical Hilbert space  $\mathcal{H}_{phys}^{(stand)}$  with the elements  $\Phi$  marked, for our present purposes, by the “curly” ket symbols  $|\Phi\rangle$ .

All the textbook [5] wisdom applies: one can employ the standard Dirac's notation and work, at least formally, with the current spectral representation of the Hamiltonian,

$$h = \sum_{n=0}^{\infty} |n\rangle \langle n| E_n \quad (5)$$

keeping in mind that for our  $h = h^\dagger$  we are allowed to assume, in the simplest scenario, that the basis  $\{|n\rangle\}$  is orthogonal and complete in  $\mathcal{H}_{phys}^{(stand)}$ . Still, in a more or less purely empirical manner, people found out that there exist several different mappings of the original and exact “complicated”  $h$  on its various “simpler” (though formally equivalent) versions  $H$ .

In a way exemplified by the above-mentioned formula  $h = \omega H \omega^{-1}$ , the second, different, “reference” Hilbert space  $\mathcal{H} = \mathcal{H}^{(ref)}$  enters the scene. Typically, in [4], a “realistic” multinucleon Hamiltonian  $h$  was assigned a simpler, bosonic isospectral partner  $H$ . At present, we may already read about a quickly growing number of applications of this idea in several branches of physics (cf., e.g., the Carl Bender's thorough review [6] of the so called  $\mathcal{PT}$ -symmetric models in field theory).

Leaving the applications and switching to an (in principle, possible [7]) generalization of the mappings with  $\omega \rightarrow \Omega$  and  $h = \Omega H \Omega^{-1}$ , each invertible mapping  $\Omega$  seems to transform  $h$  of eq. (5) (considered as acting on a given ket vector  $|\Phi\rangle \in \mathcal{H}_{phys}^{(stand)}$ ) into  $H$ . Thus, we can perceive the latter operator as simply acting on a given ket-vector element of *another*, reference space  $\mathcal{H} = \mathcal{H}^{(ref)}$  as mentioned above,

$$|\Phi\rangle := \Omega^{-1} |\Phi\rangle \in \mathcal{H}^{(ref)}, \quad \langle\Phi| := \langle\Phi| [\Omega^{-1}]^\dagger \in [\mathcal{H}^{(ref)}]^\dagger \sim \mathcal{H}^{(ref)}. \quad (6)$$

It is worth noticing that by definition, *both* the old and new spaces are *self-dual*. At the same time they are *not* unitary equivalent since, by construction,

$$\langle\Phi|\Phi'\rangle = \langle\Phi|\Theta|\Phi'\rangle, \quad \Theta = \Omega^\dagger \Omega = \Theta^\dagger > 0. \quad (7)$$

We see that another Hilbert space has to be introduced. Here, it will be denoted by the symbol  $\mathcal{H}^{(\Theta)}$  (or simply by  $\mathcal{H}_{phys}$  without superscript). It will *share* its ket vectors with the “intermediate” space  $\mathcal{H} = \mathcal{H}^{(ref)}$  (where the inner product was  $\langle\cdot|\cdot\rangle$ ). In parallel, it will *differ* from it by the innovated definition of its linear functionals,

$$\mathcal{H}_{phys}^\dagger := \{ \langle\langle\Phi| = \langle\Phi|\Omega^\dagger \Omega \equiv \langle\Phi|\Omega \} \}. \quad (8)$$

In this sense, only the mapping between  $\mathcal{H}_{phys}^{(stand)}$  and  $\mathcal{H}_{phys}$  can be considered norm-preserving and unitary.

We arrived at the very heart of the conflict between the differing opinions concerning the notation conventions as expressed in [1] and [2]. We feel that the danger emerging from a “hidden subtlety” of the self-duality questions has simply been underestimated in the latter text. Indeed,

once you consult any textbook [5] you immediately imagine that this danger is in fact specific for all the situations where one works with more (albeit unitarily equivalent!) representations of the physical Hilbert space. Then, indeed, the (usually, trivial) correspondence between “the ket vectors” (i.e., the elements of the space) and “the bra vectors” (i.e., the linear functionals in the same space) deserves an enhanced attention.

That’s why we repeat our older recommendation [1] that the linear functionals in  $\mathcal{H}_{phys} = \mathcal{H}^{(\Theta)}$  with  $\Theta \neq I$  *should* be denoted by the *specific*, doubled Dirac’s brackets  $\langle\langle \cdot |$ .

## References

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